Diagonalizing operators over continuous fields of C^*-algebras

V. M. Manuilov

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Abstract

It is well known that in the commutative case, i.e. for $\mathcal{A} = C(X)$ being a commutative C^*-algebra, compact selfadjoint operators acting on the Hilbert C^*-module $H_{\mathcal{A}}$ (= continuous families of such operators $K(x)$, $x \in X$) cannot be diagonalized inside this module but it becomes possible if we pass to a bigger module over a bigger W^*-algebra $L^\infty(X) = \mathbf{A} \supset \mathcal{A}$ which can be obtained from $\mathcal{A}$ by completing (on bounded sets) it with respect to the weak topology in the natural representation of $\mathcal{A}$ on the Hilbert space $L^2(X)$ where the norm is defined by a finite exact trace (measure) on $\mathcal{A}$. Unlike the "eigenvectors", which have coordinates from $\mathbf{A}$, the "eigenvalues" are continuous, i.e. lie in the C^*-algebra $\mathcal{A}$.

We discuss here the non-commutative analog of this well-known fact. When we pass to non-commutative C^*-algebras the "eigenvalues" are defined not uniquely but in some cases they can also be taken from the initial C^*-algebra instead of the bigger W^*-algebra. We prove here that such is the case for some continuous fields of real rank zero C^*-algebras over a one-dimensional manifold and give an example of a C^*-algebra $\mathcal{A}$ for which the "eigenvalues" cannot be chosen from $\mathcal{A}$, i.e. are discontinuous.

The main point of the proof is connected with a problem on almost commuting operators. We prove that for some C^*-algebras (including the matrix ones) if $h \in \mathcal{A}$ is a selfadjoint, $u \in \mathcal{A}$ is a unitary and if the norm of their commutant $[u, h]$ is small enough then one can connect u with the unity by a path $u(t)$ so that the norm of the commutant $[u(t), h]$ would be also small along this path.

0 Introduction

Let X be a locally compact Hausdorff space and let $\{A(x), x \in X\}$ be a family of unital C^*-algebras with exact finite traces τ_x, $\tau_x(1_x) = 1$. Denote by $\prod_{x\in X} A(x)$ the set of functions $a = a(x)$ defined on X and such that $a(x) \in A(x)$ for any $x \in X$.

Definition 0.1 *Let $\mathcal{A} \subset \prod_{x\in X} A(x)$ be a subset with the following properties:*

i) *$\mathcal{A}$ is a $*$-subalgebra in $\prod_{x\in X} A(x)$,*

ii) *for any $x \in X$ the set $\{a(x), a \in \mathcal{A}$ is dense in the algebra $A(x)$,*

iii) *for any $a \in \mathcal{A}$ the function $x \longmapsto \|a(x)\|$ is continuous,*

iv) *let $a \in \prod_{x\in X} A(x)$; if for any $x \in X$ and for any $\varepsilon > 0$ one can find such $a' \in \mathcal{A}$ that $\|a(x) - a'(x)\| < \varepsilon$ in some neighborhood of the point x, then one has $a \in \mathcal{A}$,*

v) *for any $a \in \mathcal{A}$ the function $x \longmapsto \tau_x(a(x))$ is continuous.*

Then the triple $(A(x), X, \mathcal{A})$ is called a continuous field of tracial C^*-algebras.

Notice that the first four properties of this definition give the standard definition of a continuous field of C^*-algebras [4]. It is known that $\mathcal{A}$ is a C^*-algebra. Writing in the present paper for shortness $\|a(x)\|$ we mean the norm of the element $a = a(x) \in \mathcal{A}$, i.e. $\sup_x \|a(x)\|$. If $V \subset X$ is a closed subset then we denote by $\mathcal{A}|_V$ the restriction of this algebra to the subspace V.

It is obvious that $\tau : a \longmapsto \tau_x(a(x))$ is a trace on $\mathcal{A}$ taking values in the $C(X) \subset \mathcal{A}$. This trace defines an inner product on $A(x)$,

$$(b_1, b_2)_{\tau_x} = \tau_x(b_1^* b_2), \qquad \|b\|^2_{\tau_x} = (b, b)_{\tau_x},$$

and a $C(X)$-valued inner product on $\mathcal{A}$,

$$(b_1(x), b_2(x))_\tau = \tau_x(b_1^*(x) b_2(x)) \in C(X) \tag{0.1}$$

with the norm

$$\|b(x)\|_\tau = \sup_x (b(x), b(x))_\tau^{1/2} = \sup_x \|b(x)\|_{\tau_x}.$$

Let $H(x) = L^2(A(x))$ be the completion of $A(x)$ with respect to the norm $\|\cdot\|_{\tau_x}$. Then the algebra $A(x)$ is (exactly) represented on $H(x)$ and we can pass to the corresponding W^*-algebra $B(x) = L^\infty(A(x))$, $A(x) \subset B(x) \subset H(x)$.

Let now dx be a σ-finite Borel measure on X. Notice that the function (0.1) is continuous, so one can put

$$(b_1, b_2)_\tau = \int_X (b_1(x), b_2(x))_{\tau_x}\, dx, \qquad \|b\|^2_\tau = (b, b)_\tau. \tag{0.2}$$

Denote the completion of the algebra $\mathcal{A}$ with respect to this norm by $H = L^2(\mathcal{A})$. The algebra $\mathcal{A}$ is (exactly) represented on H and the corresponding W^*-algebra we denote by $\mathbf{A} = L^\infty(\mathcal{A}) \supset \mathcal{A}$. One can see that

$$\mathbf{A} = \int_X^\oplus B(x)\, dx.$$

From the exactness and finiteness of the trace defined by τ_x on $\mathbf{A}$ and taking values in $L^\infty(X)$ it follows that the algebra $\mathbf{A}$ is a finite W^*-algebra with the finite exact trace $\overline{\tau} = \int_X \tau_x \, dx$.

Further on we will deal with the case when X is an interval or a circle. In this case if X is divided by points $\{x_k\}$ into smaller intervals $D_k = [x_k; x_{k+1}]$ and if $a_k(x) \in \mathcal{A}|_{D_k}$ is a continuous field for every k then we call the set $a(x) = \{a_k(x)\}$ a *piecewise continuous* field on X. Such piecewise continuous fields obviously belong to the W^*-algebra $\mathbf{A}$. The distance from such field to the C^*-algebra $\mathcal{A}$ is given by the formula

$$\operatorname{dist}(a(x), \mathcal{A}) = \sup_k \{\|a_k(x_k) - a_{k+1}(x_k)\|\}.$$

For shortness sake we write $K_1(x) \sim K_2(x)$ for two piecewise continuous fields $K_1(x)$ and $K_2(x)$ if there exists a piecewise continuous unitary $u(x)$ such that $u^*(x)K_1(x)u(x) = K_2(x)$.

The present paper is organized as follows. In the next section we discuss the notion of diagonalizability for operators acting on Hilbert C^*-modules.

In the section 2 we discuss a problem on almost commuting operators and prove for some C^*-algebras (including the matrix ones) that if $h \in A$ is a selfadjoint, $u \in A$ is a unitary and the norm of their commutant $[u, h]$ is small enough then one can connect u with the unity by a path $u(t)$ so that the norm of $[u(t), h]$ would be small along this path. We use it in the proof of the main theorem but it is also of independent interest in view of topological applications [6]. The estimate which we give for the norm of commutant is not the best one but it is sufficient for our purpose.

The section 3 contains the proof of the diagonalizability of selfadjoint compact operators on Hilbert C^*-modules over some continuous fields of tracial C^*-algebras of real rank zero over one-dimensional manifolds. Unfortunately our method cannot be applied to the case of arbitrary dimension of the base, though we suppose that the result is still true for them. The restrictions which we demand for the fiber C^*-algebras are far from necessary and can be weakened in different ways. In the last section we give an example of a C^*-algebra which does not allow continuous diagonalization.

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1 Diagonalizing operators on Hilbert C^*-modules

Let $\mathcal{A}$ be a C^*-algebra and let $H_{\mathcal{A}}$ be a right Hilbert $\mathcal{A}$-module of sequences $a = (a_k)$, $a_k \in \mathcal{A}$, $k \in \mathbf{N}$ such that the series $\sum a_k^* a_k$ converges in $\mathcal{A}$ in norm with the standard basis $\{e_k\}$ and let $L_n(\mathcal{A}) \subset H_{\mathcal{A}}$ be a submodule generated by the first n elements $e_1, \ldots, e_n$ of the basis. An inner $\mathcal{A}$-valued product on module $H_{\mathcal{A}}$ is given by $\langle x, y\rangle = \sum x_k^* y_k$ for $x, y \in \mathcal{A}$. Our standard references for the theory of Hilbert C^*-modules and operators on them are the papers [19],[20],[21],[10],[13],[12] and the book [11].

By $H_{\mathcal{A}}^* = \mathrm{Hom}_{\mathcal{A}}(H_{\mathcal{A}}; \mathcal{A})$ we denote the $\mathcal{A}$-module dual to $H_{\mathcal{A}}$ consisting of all bounded $\mathcal{A}$-linear $\mathcal{A}$-valued maps on $H_{\mathcal{A}}$. Remember that the module $L_n(\mathcal{A})$ is autodual, i.e. $L_n^*(\mathcal{A}) = L_n(\mathcal{A})$. A bounded operator $K : H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ is called compact [10] [13], if it possesses an adjoint operator $K^* : H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ and lies in the norm closure of the linear span of operators of the form

$$\theta_{x,y}, \quad \theta_{x,y}(z) = x\langle y, z\rangle, \quad x, y, z \in H_{\mathcal{A}}.$$

We call a compact operator K *strictly positive*, if the operator $\langle Kx, x\rangle$ is positive in $\mathcal{A}$ and if the spectral projection corresponding to the zero point of spectrum of K is zero. We will use the well-known fact [20] that in the case when $\mathcal{A}$ is a W^*-algebra the inner product can be naturally extended to the dual module $H_{\mathcal{A}}^*$. Let $M_{\mathcal{A}}$ denote either $L_n(\mathcal{A})$ or $H_{\mathcal{A}}$.

Definition 1.1 *Let $\mathbf{A}$ be a W^*-algebra. We call a selfadjoint operator K on the $\mathbf{A}$-module $M_{\mathbf{A}}$ diagonalizable if there exist a set $\{x_i\}$ of elements in $M_{\mathbf{A}}^*$ and a set of operators $\lambda_i \in \mathbf{A}$ such that*

i) $\{x_i\}$ is orthonormal, i.e. $\langle x_i, x_j\rangle = \delta_{ij}$,

ii) the module $M_{\mathbf{A}}^*$ coincides with the $\mathbf{A}$-module $\mathcal{M}^*$ dual to the module $\mathcal{M}$ generated by the set $\{x_i\}$,

iii) $Kx_i = x_i\lambda_i$,

iv) for any unitaries u_i, $u_{i+1} \in \mathbf{A}$ we have an operator inequality

$$u_i^* \lambda_i u_i \geq u_{i+1}^* \lambda_{i+1} u_{i+1}. \tag{1.1}$$

We call the elements x_i "*eigenvectors*" and the operators λ_i "*eigenvalues*" for the operator K. It should be noticed that the "eigenvectors" and "eigenvalues" are defined not uniquely. The condition (1.1) means that the notion of diagonalization includes the natural ordering of the "eigenvalues".

The problem of diagonalizing operators in Hilbert modules was initiated by R. V. Kadison in [9] who proved that a selfadjoint operator on the module $L_n(\mathbf{A})$ over a W^*-algebra $\mathbf{A}$ can be diagonalized. Further this problem was studied in different settings in [8],[18],[26],[7] etc. In [14],[15] we have proved the following

Theorem 1.2 *If $\mathbf{A}$ is a finite W^*-algebra with a normal exact finite trace then a compact strictly positive operator K on the module $H_{\mathbf{A}}$ can be diagonalized on $H^*_{\mathbf{A}}$ and its "eigenvalues" are defined uniquely up to the unitary equivalence.*

In the paper [16] it was proved also that the "eigenvalues" depend continuously on the compact operators, namely

Theorem 1.3 *If $K_r : H_{\mathbf{A}} \longrightarrow H_{\mathbf{A}}$, $r = 1, 2$ are compact strictly positive operators and if $\|K_1 - K_2\| < \varepsilon$ then*

i) one can find a unitary U acting on $H^*_{\mathbf{A}}$ such that it maps the "eigenvectors" of K_2 to the "eigenvectors" of K_1 and $\|U^* K_1 U - K_2\| < \varepsilon$,

ii) "eigenvalues" $\{\lambda_i^{(r)}\}$ of operators K_r $(r = 1, 2)$ can be chosen in such a way that $\left\|\lambda_i^{(1)} - \lambda_i^{(2)}\right\| < \varepsilon$.

It is well known that in the commutative case, i.e. for $\mathcal{A} = C(X)$ being a commutative C^*-algebra, compact operators cannot be diagonalized inside $H_{\mathcal{A}}$ but it becomes possible if we pass to a bigger module over a bigger W^*-algebra $L^\infty(X) = \mathbf{A} \supset \mathcal{A}$ which can be obtained from $\mathcal{A}$ by completing (on bounded sets) it with respect to the weak topology in the natural representation of $\mathcal{A}$ on the Hilbert space $H = L^2(\mathcal{A})$ with the new (integral) norm defined by a finite exact trace (measure) on $\mathcal{A}$. Unlike the "eigenvectors" the "eigenvalues" are continuous, i.e. lie in the C^*-algebra $\mathcal{A}$. It leads us to the following definition.

Let $\mathcal{A}$ be a C^*-algebra with a finite exact trace τ on it. Let $(a, b)_\tau = \tau(a^* b)$, $a, b \in \mathcal{A}$, be a non-degenerate inner product on $\mathcal{A}$, $\|\cdot\|_\tau = (\cdot, \cdot)_\tau^{1/2}$ be the norm defined by the trace τ on $\mathcal{A}$. Completing $\mathcal{A}$ with respect to this norm we obtain a Hilbert space $L^2(\mathcal{A})$ and an exact representation of $\mathcal{A}$ on this space. Let $\mathbf{A} = L^\infty(\mathcal{A})$ be the corresponding finite tracial W^*-algebra containing $\mathcal{A}$ as a weakly dense subalgebra. Let K be a compact strictly positive operator on $H_{\mathcal{A}}$. We can naturally extend K to the bigger module $H^*_{\mathbf{A}}$ where it will remain compact and strictly positive and by the theorem 1.2 it can be diagonalized in this module.

Definition 1.4 *We call a C^*-algebra $\mathcal{A}$ admitting weak diagonalization if the diagonal entries (after diagonalization on $H^*_{\mathbf{A}}$) for any compact strictly positive operator K on $H_{\mathcal{A}}$ can be taken from $\mathcal{A}$ instead of $\mathbf{A}$.*

Problem. Describe the class of tracial C^*-algebras admitting weak diagonalization.

Notice that if a C^*-algebra has the weak diagonalization property then every selfadjoint finite rank operator $K \in M_n \otimes \mathcal{A}$ also can be diagonalized over the W^*-algebra $\mathbf{A}$ with "eigenvalues" being from $\mathcal{A}$.

Recall that real rank zero ($RR(\mathcal{A}) = 0$) means [1] that every selfadjoint operator in $\mathcal{A}$ can be approximated by operators with finite spectrum, i.e. having the form

$\sum \alpha_i p_i$, where $p_i \in \mathcal{A}$ are selfadjoint mutually orthogonal projections and $\alpha_i \in \mathbf{R}$. By [1] we have in this case also $RR(\mathrm{End}_{\mathcal{A}}(L_n(\mathcal{A}))) = 0$. In the paper [16] it was shown that besides the commutative C^*-algebras this class contains also real rank zero tracial C^*-algebras with the following property:

(∗) for any two projections $p, q \in \mathcal{A}$ there exist in $\mathcal{A}$ equivalent (in $\mathcal{A}$) projections $r_p \sim r_q$, $r_p \le p$, $r_q \le q$ such that $T(r_p) = T(r_q) = \min\{T(p)(z), T(q)(z)\}$, $z \in Z$ where $L^\infty(Z)$ is the center of the W^*-algebra and T is the standard center-valued trace on $L^\infty(\mathcal{A})$.

It means that $K_0(\mathcal{A})$ is a sublattice in $K_0(L^\infty(\mathcal{A}))$. In the case when the algebra $L^\infty(\mathcal{A})$ is a (finite) factor then the property (∗) means that the map $K_0(\mathcal{A}) \longrightarrow K_0(L^\infty(\mathcal{A}))$ is a monomorphism. Besides finite factors this class of algebras contains the irrational rotation C^*-algebras [3] and the Bunce-Deddens algebras [2].

2 On almost commuting operators

The following proposition concerning almost commuting operators in some C^*-algebras (including matrix algebras) will be used to diagonalize continuous fields of operators. Remember that $tsr(A) = 1$ means that the invertible elements are dense in A.

Proposition 2.1 *Let A be a C^*-algebra with properties*

i) $RR(A) = 0$ and $tsr(A) = 1$;

ii) for every projection $p \in A$ the unitary group of the C^*-algebra pAp is connected.

Let $h \in A$ be a self-adjoint and let $u \in A$ be a unitary such that

$$\|u^* h u - h\| < \delta. \tag{2.1}$$

Then there exists a constant C depending only on $\|h\|$ and a path $u(t)$ connecting u with 1 such that for small enough δ one has for all t

$$\|u^*(t) h u(t) - h\| < C \sqrt[4]{\delta}.$$

Proof. As $RR(A) = 0$ we can assume without loss of generality that the operator h is a linear combination of mutually orthogonal projections, $h = \sum_{i=1}^n \lambda_i p_i \in A$ with real eigenvalues λ_i. We can also assume that these eigenvalues are ordered, i.e. $\lambda_1 > \ldots > \lambda_n$. Divide the segment $[\lambda_n; \lambda_1]$ into smaller segments of the length $\sqrt[4]{\delta}$ and denote those segments which contain at least one eigenvalue λ_i by Δ_k. Then the number m of such segments is not bigger then $(\lambda_1 - \lambda_n)/\sqrt[4]{\delta} + 1$ and the following properties hold:

i) if $\lambda_i, \lambda_j \in \Delta_k$ then $|\lambda_i - \lambda_j| < \sqrt[4]{\delta}$;

ii) if $\lambda_i \in \Delta_{k-1}$ and $\lambda_j \in \Delta_{k+1}$ then $|\lambda_i - \lambda_j| \geq \sqrt[4]{\delta}$.

Let μ_k be the middle points of the segments Δ_k. Then if $\mu_{k+1} - \mu_k > \sqrt[4]{\delta}$ then the spectrum of h has a lacuna of the length not less than $\sqrt[4]{\delta}$.
Denote the spectral projections of h corresponding to the segments Δ_k by q_k, $q_1 + \ldots + q_m = 1$. Then A as A-module can be decomposed into a direct sum corresponding to the above projections, $A = \oplus_{k=1}^m q_k A$ and we will represent elements of the algebra A as matrices with regards to this decomposition: $h = \mathrm{diag}(\{h_i\})$ for $h_i = q_i h q_i$ and $u = (u_{ij})$ for $u_{ij} = q_i u q_j$.

Notice that if $m = 1$, i.e. all eigenvalues of h differ from each other not more than by $\sqrt[4]{\delta}$ then we can take any path $u(t)$ connecting u with 1 because in this case there exists a number μ such that $\|h - \mu\| < \sqrt[4]{\delta}/2$, hence

$$\|v^* h v - h\| \leq \|v^* h v - v^* \mu v\| + \|\mu - h\| < 2\frac{\sqrt[4]{\delta}}{2} = \sqrt[4]{\delta}$$

for any unitary v, so further we can assume that $m > 1$.

Divide once more the spectrum of the operator h into smaller (than Δ_k) segments $\overline{\Delta}_s$ of length δ. Let $\overline{\lambda}_s$ be the middle points of the segments $\overline{\Delta}_s$ and let $\overline{p}_s$ be the spectral projections of h corresponding to the segments $\overline{\Delta}_s$. Then put

$$\overline{h} = \sum_s \overline{\lambda}_s \overline{p}_s.$$

Obviously $\left\|h - \overline{h}\right\| < \delta/2$, hence in view of (2.1)

$$\left\|u^* \overline{h} u - \overline{u}\right\| \leq \left\|u^* \overline{h} u - u^* h u\right\| + \|u^* h u - h\| + \left\|h - \overline{h}\right\| < 2\delta. \tag{2.2}$$

Let $A = \oplus_s \overline{p}_s A$ be the decomposition of A corresponding to the spectral projections of h. It is a sub-decomposition of $\oplus_{k=1}^m q_k A$ and the matrix u can be written as $u = (v_{kl})$, $v_{kl} = \overline{p}_k u \overline{p}_l$ and the matrix entries u_{ij} can be viewed as blocks of elements v_{kl}. Denote by N the number of columns of the matrix (v_{kl}). Then one has

$$N < \frac{2\|h\|}{\delta} + 1 \tag{2.3}$$

Now we turn to construction of the homotopy $u \sim 1$. We begin with a non-unitary path $u(t)$ which would lie not far from the set of unitaries $U = U(A)$.

First step of homotopy

For a matrix $a = (a_{ij})$ decomposed with respect to $\oplus_{k=1}^m q_k A$ we denote by $d_0(a)$ its main diagonal $\mathrm{diag}(\{a_{ii}\})$, by $d_k(a)$ denote a diagonal lying k lines above (or

below if k is negative) the main diagonal. We start by proving that the matrix $u = (u_{ij})$ is "almost" three-diagonal.
Let k_i and k_j be the numbers of the segments Δ_k which contain the eigenvalues $\overline{\lambda}_i$ and $\overline{\lambda}_j$ of $\overline{h}$ respectively; $\overline{\lambda}_i \in \Delta_{k_i}$, $\overline{\lambda}_j \in \Delta_{k_j}$. Define a three-diagonal matrix $d(a)$ by the following way:

i) if $k_j \geq k_i + 2$ or $k_j \leq k_i - 2$ then put $(d(a))_{ij} = 0$,

ii) if $k_j = k_i \pm 1$ and $|\mu_{k_j} - \mu_{k_i}| > \sqrt[4]{\delta}$ then put also $(d(a))_{ij} = 0$,

iii) otherwise put $(d(a))_{ij} = a_{ij}$.

Lemma 2.2 *It follows from (2.1) that for small enough δ one has*

$$\|u - d(u)\| < 4\,\|h\|^{1/2}\,\sqrt[4]{\delta}.$$

Proof. Consider the matrix $(\overline{h}u - u\overline{h})(\overline{h}u - u\overline{h})^*$. From the inequality (2.2) it follows that the norm of this matrix is less then $4\delta^2$, hence the norm of any element of this matrix is also less than $4\delta^2$. So, as $\overline{\lambda}_j$ commute with v_{kj}, we obtain for every i

$$\left\|\sum_{j=1}^{N}(\overline{\lambda}_i - \overline{\lambda}_j)^2 v_{ij}v_{ij}^*\right\| < 4\delta^2. \tag{2.4}$$

Let $\overline{\lambda}_i \in \Delta_{k_i}$, $\overline{\lambda}_j \in \Delta_{k_j}$. As all the summands in (2.4) are positive, so ignoring some of them we will not increase the norm of the sum, hence

$$\left\|\sum\nolimits_j'(\overline{\lambda}_i - \overline{\lambda}_j)^2 v_{ij}v_{ij}^*\right\| < 4\delta^2$$

where the sum $\sum'$ is taken for those j for which either $|k_j - k_i| \geq 2$ or $|k_j - k_i| = 1$ and $|\mu_{k_j} - \mu_{k_i}| > \sqrt[4]{\delta}$, i.e. we throw away those v_{ij} for which $(u - d(u))_{ij} = 0$. As in the sum $\sum'$ we have $|\overline{\lambda}_k - \overline{\lambda}_j| \geq \sqrt[4]{\delta}$, so

$$4\delta^2 > \left\|\sum\nolimits_j'(\overline{\lambda}_i - \overline{\lambda}_j)^2 v_{ij}v_{ij}^*\right\| \geq \sqrt{\delta}\left\|\sum\nolimits_i' v_{ij}v_{ij}^*\right\|,$$

hence

$$\left\|\sum\nolimits_j' v_{ij}v_{ij}^*\right\| < 4\delta^{3/2}. \tag{2.5}$$

Lemma 2.3 *Let $a = (a_{ij})$, $a_{ij} \in A$ be a $N \times N$ matrix such that for every i one has $\left\|\sum_j a_{ij}a_{ij}^*\right\| < \varepsilon^2$. Then $\|a\| < \varepsilon\sqrt{N}$.*

Proof. Take $\xi = (\xi_k)$, $k = 1, \ldots, N$, $\xi_i \in A$. Then using the generalized Cauchi-Schwartz inequality [11] we get

$$\begin{aligned} \|a\xi\|^2 &= \left\|\sum_{ijk} \xi_j^* a_{ij}^* a_{ik} \xi_k\right\| \leq \sum_i \left\|\sum_{kj} \xi_j^* a_{ij}^* a_{ik} \xi_k\right\| \\ &\leq \sum_i \left\|\sum_k a_{ik} a_{ik}^*\right\| \cdot \left\|\sum_k \xi_k^* \xi_k\right\| < N\varepsilon^2 \|\xi\|^2, \end{aligned}$$

hence we have $\|a\xi\| < \sqrt{N}\varepsilon \|\xi\|$. •

The sums $\sum'$ correspond to the blocks lying in the matrix $u - d(u)$. Now in view of (2.3) and (2.5) from the lemma 2.3 (for $\varepsilon = 2\delta^{3/4}$) we get for $\delta < \|h\|$

$$\|u - d(u)\| < \sqrt{N}\ 2\delta^{3/4} \leq 2\|h\|^{1/2}\delta^{-1/2} 2\delta^{3/4} = 4\|h\|^{1/2}\sqrt[4]{\delta}$$

which ends the proof of the lemma 2.2. •

Define the path $u(t)$ by the formula

$$u(t) = (u - d(u))(1 - t) + d(u).$$

Then $u(0) = u$, $u(1) = d(u)$ and for every $t \in [0, 1]$ by the lemma 2.2 we have $\operatorname{dist}(u(t), U) < 4\|h\|^{1/2}\sqrt[4]{\delta}$. Estimate the commutator norm:

$$\begin{aligned} \left\|u(t)\overline{h} - \overline{h}u(t)\right\| &\leq \left\|u\overline{h} - \overline{h}u\right\| + \left\|(u - d(u))\overline{h} - \overline{h}(u - d(u))\right\| \\ &< 2\delta + 2\|h\|\,\|u - d(u)\| < 2\delta + 2\|h\|\,4\|h\|^{1/2}\sqrt[4]{\delta}. \end{aligned}$$

Second step of homotopy

We should connect the matrix $d(u)$ with a diagonal one. To do so we need the following

Lemma 2.4 (making matrices almost upper triangular). *Let*

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a matrix in $q_{j_1}A \oplus q_{j_2}A$. Then for any $\varepsilon > 0$ there exists a unitary path $v(t)$ such that $v(0) = 1$ and

$$v(1) \cdot a = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$$

with $\|a'_{21}\| < \varepsilon$.

Proof. As by assumption $tsr(A) = 1$, so for any $\varepsilon > 0$ we can find an invertible element $\overline{a}_{11}$ such that $\|\overline{a}_{11} - a_{11}\| < \varepsilon$. Put $\alpha = a_{21}(\overline{a}_{11})^{-1}$. Put further

$$v(t) = \begin{pmatrix} (1+t^2\alpha^*\alpha)^{-1/2} & (1+t^2\alpha^*\alpha)^{-1/2} \cdot t\alpha^* \\ -(1+t^2\alpha\alpha^*)^{-1/2} \cdot t\alpha & (1+t^2\alpha\alpha^*)^{-1/2} \end{pmatrix}.$$

It can be easily seen that $v^*(t)v(t) = v(t)v^*(t) = 1$ and $v(1)$ lies in the unitary group in the path component of 1. Estimate the entry a'_{21} in the product $v(1)a$:

$$\begin{aligned} \|a'_{21}\| &= \left\| -(1+\alpha\alpha^*)^{-1/2}\alpha a_{11} + (1+\alpha\alpha^*)^{-1/2} a_{21} \right\| \\ &= \left\| (1+\alpha\alpha^*)^{-1/2}(\alpha a_{11} - \alpha\overline{a}_{11}) \right\| \leq \left\| (1+\alpha\alpha^*)^{-1/2}\alpha \right\| \cdot \|a_{11} - \overline{a}_{11}\| \\ &< \varepsilon \cdot \left\| (1+\alpha\alpha^*)^{-1/2}\alpha \right\| = \varepsilon \cdot \|f(\beta)\|^{1/2} \leq \varepsilon \end{aligned}$$

where $\beta = \alpha\alpha^*$ and $f(\lambda) = \frac{\lambda}{1+\lambda} \leq 1$. •

Consider the element u_{21} of the matrix u. If the segments Δ_1 and Δ_2 are separated then the element u_{21} is already small enough and the element $(d(u))_{21}$ is already zero. Put then $v_1 = 1$. If these segments are not separated then by the lemma 2.4 we can find in the module $q_1 A \oplus q_2 A$ a unitary path $v_1(t)$ with $v_1(0) = 1$ and

$$v_1(1) = \begin{pmatrix} v_{11}^{(1)} & v_{12}^{(1)} \\ v_{21}^{(1)} & v_{22}^{(1)} \end{pmatrix}$$

such that

$$\begin{pmatrix} v_{11}^{(1)} & v_{12}^{(1)} & \vdots & \\ v_{21}^{(1)} & v_{22}^{(1)} & \vdots & \\ \cdots & \cdots & \cdots & \\ & & \vdots & E \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & & \\ u_{21} & u_{22} & u_{23} & \\ & \cdots & \cdots & \cdots \\ & & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} u_{11}^{(1)} & u_{12}^{(1)} & u_{13}^{(1)} & \\ u_{21}^{(1)} & u_{22}^{(1)} & u_{23}^{(1)} & \\ & \cdots & \cdots & \cdots \\ & & \cdots & \cdots \end{pmatrix}$$

with $\left\| u_{21}^{(1)} \right\| < \varepsilon$. Here E stands for a unit matrix and empty places stand for zeros. Notice that then the third and the following strings of the matrix u do not change. So we get a path $u(t) = v_1(t)d(u)$ and as $v_1(t)$ is a unitary, so

$$\operatorname{dist}(u(t), U) < 4\|h\|^{1/2} \sqrt[4]{\delta}. \tag{2.6}$$

As in the lemma 2.4 the number ε can be taken arbitrarily small, so if we write zero instead of the element $u_{21}^{(1)}$, the estimate (2.6) would remain valid.

Turn now to the element $u_{32} = u_{32}^{(1)}$. If the segments Δ_2 and Δ_3 are separated then this element is already small enough. Put then $v_2 = 1$. In the other case we can find a unitary path $v_2(t)$ with $v_2(0) = 1$ and

$$v_2(1) = \begin{pmatrix} v_{22}^{(2)} & v_{23}^{(2)} \\ v_{32}^{(2)} & v_{33}^{(2)} \end{pmatrix}$$

in the module $q_2A \oplus q_3A$ such that

$$
\begin{pmatrix}
1 & & & & \\
& v_{22}^{(2)} & v_{23}^{(2)} & \vdots & \\
& v_{32}^{(2)} & v_{33}^{(2)} & \vdots & \\
& \cdots & \cdots & \cdots & \\
& & & \vdots & E
\end{pmatrix}
\cdot
\begin{pmatrix}
u_{11}^{(1)} & u_{12}^{(1)} & u_{13}^{(1)} & & \\
& u_{22}^{(1)} & u_{23}^{(1)} & & \\
& u_{32}^{(1)} & u_{33}^{(1)} & u_{34}^{(1)} & \\
& & \cdots & \cdots & \cdots \\
& & & \cdots & \cdots
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
u_{11}^{(2)} & u_{12}^{(2)} & u_{13}^{(2)} & & \\
& u_{22}^{(2)} & u_{23}^{(2)} & u_{24}^{(2)} & \\
& u_{32}^{(2)} & u_{33}^{(2)} & u_{34}^{(2)} & \\
& & \cdots & \cdots & \cdots \\
& & & \cdots & \cdots
\end{pmatrix}
$$

with $\left\| u_{32}^{(2)} \right\| < \varepsilon$. Again we can write zero instead of $u_{32}^{(2)}$. Repeating this procedure we obtain a unitary path

$$v(t) = v_{m-1}(t) \cdot \ldots \cdot v_2(t) \cdot v_1(t)$$

connecting the unity with the unitary

$$v = v(m-1) = v_{m-1}(m-1) \cdot \ldots \cdot v_2(2) \cdot v_1(1).$$

Denote $v(t) \cdot d(u)$ by $\overline{u}(t)$ and $v \cdot d(u)$ by $\overline{u}$. Notice that

$$\operatorname{dist}(\overline{u}(t), U) < 4 \|h\|^{1/2} \sqrt[4]{\delta} \tag{2.7}$$

for all t. Hence we have the estimate $\|\overline{u}_{ij}(t)\| < 4 \|h\|^{1/2} \sqrt[4]{\delta}$ for every i, j. Notice also that for every t the path $\overline{u}(t)$ lies in the four-diagonal matrices of the form

$$
\overline{u}(t) =
\begin{pmatrix}
\overline{u}_{11} & \overline{u}_{12} & \overline{u}_{13} & & & \\
\overline{u}_{21} & \overline{u}_{22} & \overline{u}_{23} & \overline{u}_{24} & & \\
& \overline{u}_{32} & \overline{u}_{33} & \overline{u}_{34} & \overline{u}_{35} & \\
& & \cdots & \cdots & \cdots & \cdots \\
& & & \cdots & \cdots & \cdots \\
& & & & \cdots & \cdots
\end{pmatrix}
$$

therefore we can easily estimate the commutator norm along this path. To do so we should deal with the operator $h' = \sum \mu_k q_k$, $\|h - h'\| < \frac{1}{2}\sqrt[4]{\delta}$. Then

$$
\begin{aligned}
\|\overline{u}(t)h - h\overline{u}(t)\| &< \|[\overline{u}(t), h']\| + \sqrt[4]{\delta} \\
&\leq \|[d_{-1}(\overline{u}(t)), h']\| + \|[d_1(\overline{u}(t)), h']\| + \|[d_2(\overline{u}(t)), h']\| + \sqrt[4]{\delta} \\
&\leq \sup_{i,j,k} \|\overline{u}_{ij}(t)\| \left(2|\mu_k - \mu_{k+1}| + |\mu_k - \mu_{k+2}|\right) + \sqrt[4]{\delta} \\
&< 6\sqrt[4]{\delta}
\end{aligned} \tag{2.8}
$$

for small enough δ. The resulting matrix $\overline{u}$ is three-diagonal and upper triangular:

$$\overline{u} = \begin{pmatrix} \overline{u}_{11} & \overline{u}_{12} & \overline{u}_{13} & & \\ & \overline{u}_{22} & \overline{u}_{23} & \overline{u}_{24} & \\ & & \overline{u}_{33} & \overline{u}_{34} & \overline{u}_{35} \\ & & & \cdots & \cdots \\ & & & & \cdots \end{pmatrix}.$$

Third step of homotopy

It follows from (2.7) that the matrix $\overline{u}$ is invertible and there exists a unitary w such that

$$\|\overline{u} - w\| < 4 \|h\|^{1/2} \sqrt[4]{\delta}.$$

Therefore when δ is small enough we have

$$\left\|(\overline{u})^{-1} - w^*\right\| < 6 \|h\|^{1/2} \sqrt[4]{\delta}.$$

Therefore we get the estimate

$$\left\|(\overline{u})^{-1} - (\overline{u})^*\right\| < 10 \|h\|^{1/2} \sqrt[4]{\delta},$$

hence

$$\|d_1(\overline{u})\| < 10 \|h\|^{1/2} \sqrt[4]{\delta} \quad \text{and} \quad \|d_2(\overline{u})\| < 10 \|h\|^{1/2} \sqrt[4]{\delta}.$$

Therefore

$$\mathrm{dist}(d_0(\overline{u}), U) < \mathrm{dist}(\overline{u}, U) + \|d_1(\overline{u})\| + \|d_2(\overline{u})\| < 24 \|h\|^{1/2} \sqrt[4]{\delta}. \tag{2.9}$$

Connect the matrices $\overline{u}$ and $d_0(\overline{u})$ by a linear path $\overline{\overline{u}}(t)$. Then along this path one has

$$\mathrm{dist}(\overline{\overline{u}}(t), U) < 24 \|h\|^{1/2} \sqrt[4]{\delta} \tag{2.10}$$

and from (2.8) we see that

$$\|\overline{\overline{u}}(t)h - h\overline{\overline{u}}(t)\| < 6 \|h\|^{1/2} \sqrt[4]{\delta}.$$

Fourth step of homotopy

It follows from (2.9) that for small enough δ the diagonal matrix $d_0(\overline{u})$ consists of invertible elements being close to unitaries and it follows from (2.10) that for every $\overline{u}_{ii}$ one can find a unitary w_i such that $\|\overline{u}_{ii} - w_i\| < 24 \|h\|^{1/2} \sqrt[4]{\delta}$. Take a linear path connecting the matrices $\overline{u}$ and $\mathrm{diag}\{w_i\}$. Then this path also lies close to the unitary group U. Then we connect the matrix $\mathrm{diag}\{w_i\}$ with unity. The last path lies in U. Let $\tilde{u}(t)$ be the path of the fourth step of homotopy.

Notice that it lies within diagonal matrices, hence the commutator norm along this path is small:

$$\|\widetilde{u}(t)h - h\widetilde{u}(t)\| \leq \|\widetilde{u}(t)h' - h'\widetilde{u}(t)\| + \sqrt[4]{\delta} = \sqrt[4]{\delta}.$$

Consider now all four steps of constructed homotopy. We see that along the whole path $u'(t)$ connecting u with unity we have

$$\mathrm{dist}(u'(t), U) < 24 \|h\|^{1/2} \sqrt[4]{\delta}$$

and

$$\|u'(t)h - hu'(t)\| < 6\sqrt[4]{\delta}.$$

Therefore there exists a unitary path $u(t)$ connecting u with unity such that

$$\|u(t) - u'(t)\| < 48 \|h\|^{1/2} \sqrt[4]{\delta},$$

hence we get the estimate

$$\begin{aligned} \|u(t)h - hu(t)\| &< \|u'(t)h - hu'(t)\| + 2\|h\| \|u(t) - u'(t)\| \\ &< 6\sqrt[4]{\delta} + 96 \|h\|^{3/2} \sqrt[4]{\delta} = C\sqrt[4]{\delta} \end{aligned}$$

which proves the proposition. •

Remark that the class of C^*-algebras satisfying conditions of the proposition 2.1 includes the finite W^*-algebras and particularly the matrix algebras. The constant C is by no means the best one and can be improved. On the other hand I believe that the power $1/4$ for δ in our estimate cannot be made bigger. As the commutator norm does not change if we replace h by $h + \lambda$, $\lambda \in \mathbf{R}$, so the constant C really depends not on the norm of h but on the length of its spectrum.

Remark also that by the same way one can proof the similar result in the case when instead of selfadjoint h we deal with a unitary v with a lacuna in its spectrum. The last restriction is essential as otherwise there exists an obstruction [6]. The following proposition can be viewed as a first step in construction of approximately commuting homotopy between approximately commuting and commuting pairs of unitaries in the case when the obstruction of [6] is zero.

Proposition 2.5 *Let A be a C^*-algebra with properties*

i) $RR(A) = 0$ *and* $tsr(A) = 1$*;*

ii) for every projection $p \in A$ the unitary group of the C^*-algebra pAp is connected.

Let $u, v \in A$ be unitaries such that $\|u^*hu - h\| < \delta$. Suppose that the unit circle contains an arc of the length d such that the intersection of this arc with the spectrum of v is empty. Then there exists a constant C depending only on d and a path $u(t)$ connecting u with 1 such that for small enough δ and for all t one has $\|u^*(t)hu(t) - h\| < C\sqrt[4]{\delta}$.

3 Diagonalizing operators over some continuous fields of C^*-algebras

Theorem 3.1 *Let $(A(x), X, \mathcal{A})$ be a continuous field of tracial C^*-algebras over an interval or a circle. Let $(B(x), X, \mathcal{B})$ be the corresponding continuous field of W^*-algebras and let $\mathbf{A}$ be the corresponding W^*-algebra. Suppose that for every $x \in X$ the C^*-algebra possesses the following properties:*

i) $RR(A(x)) = 0$, *and* $tsr(A(x)) = 1$;

ii) for every projection $p \in A(x)$ the unitary group of the C^*-algebra $pA(x)p$ is connected;

iii) the trace τ_x on $A(x)$ is finite and exact;

iv) the map $K_0(A(x)) \longrightarrow K_0(B(x))$ induced by inclusion $A(x) \subset B(x) = L^\infty(A(x))$ is a monomorphism and $K_0(A(x))$ is a sublattice in $K_0(B(x))$.

Then the C^*-algebra $\mathcal{A} \in \mathbf{A}$ possesses the property of weak diagonalization.

Proof is based on the following lemmas.

Lemma 3.2 *Let $K_1(x)$, $K_2(x)$ be two continuous fields of selfadjoint operators with finite spectrum in the algebra $\mathcal{A}$, K_r being unitarily equivalent to* $\mathrm{diag}\{\lambda_i^{(r)}\}$, *$r = 1, 2$. If $\|K_1(x) - K_2(x)\| < \varepsilon$ then for any $x_0 \in X$ there exists a closed neighborhood W of x_0 and continuous unitary fields $u_i(x)$ on W such that*

$$\left\|\lambda_i^{(1)}(x) - u_i^*(x)\lambda_i^{(2)}(x)u_i(x)\right\| < \varepsilon. \tag{3.1}$$

Proof. By supposition we can write the operators $K_r(x)$ in the form

$$K_1(x) = \sum \alpha_m p_m(x), \quad K_2(x) = \sum \beta_j q_j(x)$$

with ordered eigenvalues and $p_m(x)$, $q_j(x)$ being the spectral projections. As $K_0(A(x_0)) \subset K_0(B(x_0))$ is a sublattice by supposition, so we can find projections $r_l(x_0) \in A(x_0)$ such that $\sum_l r_l(x_0) = 1 \in M_n \otimes A(x_0)$ and every projection $p_m(x_0)$, $q_j(x_0)$ and projections from $M_n \otimes 1$ are unitarily equivalent to sums of some $r_l(x_0)$. After renumbering the eigenvalues of $K_r(x_0)$ and admitting repeating eigenvalues we can write

$$K_1(x_0) = \sum \alpha_l r_l(x_0); \quad K_2(x_0) = \sum \beta_l r_l'(x_0); \quad r_l(x_0) \sim r_l'(x_0).$$

As in the W^*-algebra $B(x_0) = L^\infty(A(x_0))$ we have $\|K_1(x_0) - K_2(x_0)\| < \varepsilon$ so (cf. [24]) for all n we have

$$|\alpha_l - \beta_l| < \varepsilon. \tag{3.2}$$

Divide the set of projections $\{r_l(x_0)\}$ into n groups

$$\{r_1(x_0), \dots, r_{l_1}(x_0)\}; \dots ; \{r_{l_{n-1}+1}(x_0), \dots, r_{l_n}(x_0)\}$$

so that the sum of projections in each group would be unitarily equivalent to the one-dimensional projection in $M_n \otimes 1$. Then each group gives us the "eigenvalues" $\lambda_i^{(r)}(x_0)$:

$$\begin{aligned}
\lambda^{(1)}(x_0) &= \alpha_{l_{i-1}+1} r_{l_{i-1}+1}(x_0) + \ldots + \alpha_{l_i} r_{l_i}(x_0); \\
\lambda^{(2)}(x_0) &= \beta_{l_{i-1}+1} r'_{l_{i-1}+1}(x_0) + \ldots + \beta_{l_i} r'_{l_i}(x_0).
\end{aligned}$$

Take unitaries $u_i(x_0) \in A(x_0)$ such that $r'_l(x_0) = u_i^*(x_0) r_l(x_0) u_i(x_0)$ for $l_{i-1}+1 \leq l \leq l_i$. Then the estimate (3.1) follows from (3.2). •

Lemma 3.3 *Let $D = [x_k; x_{k+1}]$ be an interval in X and let $K_1(x)$, $K'_2(x) \in \mathcal{A}|_D$ be two continuous fields of selfadjoint operators with finite spectrum on D with $\|K_1(x) - K'_2(x)\| < \varepsilon$ and let $K_2(x) \in \mathcal{A}(x_k)$ be such selfadjoint operator with finite spectrum that there exists a unitary $u(x_k) \in A(x_k)$ such that*

$$\|K_2(x_k) - u^*(x_k) K'_2(x_k) u(x_k)\| < \delta \quad \text{and} \quad \|K_1(x_k) - K_2(x_k)\| < \varepsilon.$$

Then for small enough δ and ε there exists a piecewise continuous unitary field $u(x)$ such that on D one has

$$\|u^*(x) K'_2(x) u(x) - K_1(x)\| < C \sqrt[4]{2\varepsilon + \delta} \tag{3.3}$$

where C is a constant depending only on the norm of $K_1(x)$.

Proof. Let $u_t(x_k)$, $0 \leq t \leq 1$, be a path connecting $u(x_k)$ with unity in the unitary group of $A(x_k)$. By assumption we have $\|K_2(x_k) - K'_2(x_k)\| < 2\varepsilon$, hence

$$\|K'_2(x_k) - u_t^*(x_k) K'_2(x_k) u_t(x_k)\| < 2\varepsilon + \delta.$$

Then by the proposition 2.1 there exists a constant C depending only on the norm of $K_1(x_k)$ (or of $K'_2(x_k)$ as they are close) and a path $u_t(x_k)$ connecting $u(x_k)$ with unity so that

$$\|u_t^*(x_k) K'_2(x_k) u_t(x_k) - K_1(x_k)\| < C \sqrt[4]{2\varepsilon + \delta}. \tag{3.4}$$

It follows from (3.4) that there exists an interval $[x_k; x'_{k+1}]$ such that for any $x \in [x_k; x'_{k+1}]$ we still have

$$\|u_t^*(x) K'_2(x) u_t(x) - K_1(x)\| < C \sqrt[4]{2\varepsilon + \delta}.$$

Put then

$$u(x) = u_t(x) \qquad \text{where} \quad t = \frac{x'_{k+1} - x}{x'_{k+1} - x_k}.$$

It is a unitary continuous field and on $[x_k; x'_{k+1}]$ the estimate (3.3) holds. On the other hand when $x \geq x_{k+1}$ we have $u(x) = 1$, so there this estimate also holds. •

Lemma 3.4 *Let $K(x)$ be a selfadjoint operator on the module $L_n(\mathcal{A})$, $K(x) \in M_n \otimes \mathcal{A}$. Then for any $\varepsilon > 0$ there exists a piecewise continuous operator field with discrete spectrum $K'(x)$ locally being from $M_n \otimes \mathcal{A}$ such that $\|K'(x) - K(x)\| < \varepsilon$ and $K'(x)$ is diagonalizable in $L_n \otimes \mathbf{A}$ with "eigenvalues" $\lambda_i(x)$ being piecewise continuous and* $\operatorname{dist}(\lambda_i(x), \mathcal{A}) < 2\varepsilon$.

Proof. Fix $\varepsilon > 0$ and take a point $x_0 \in X$. As $RR(A(x_0)) = 0$ we can find an operator $K'(x_0) \in M_n \otimes A(x_0)$ with finite spectrum, $K'(x_0) = \sum_m \alpha_m p_m(x_0)$, $p_m(x_0)$ being its spectral projections, such that $\|K'(x_0) - K(x_0)\| < \frac{\varepsilon}{2}$. There exists a neighborhood of the point x_0 such that the projections $p_m(x_0)$ can be extended to projections $p_m(x)$ in this neighborhood. Put $K'(x) = \sum_m \alpha_m p_m(x)$. Then there exists a smaller neighborhood of x_0 where

$$\|K'(x) - K(x)\| < \varepsilon. \tag{3.5}$$

So every point of X possesses a neighborhood D_k and a continuous field on D_k such that (3.5) holds. Taking a finite covering of X by such neighborhoods we obtain a division of X by smaller intervals and a piecewise continuous field $K'(x) = \{K'_k(x)\}$, $K'_k(x) \in M_n \otimes \mathcal{A}|_{D_k}$, such that $\operatorname{dist}(K'(x), M_n \otimes \mathcal{A}) < 2\varepsilon$. Diagonalize the operator field $K'(x)$ on every interval D_k, $K'(x) = \operatorname{diag}\{\lambda_i(x)\}$. Then the fields $\lambda_i(x) = \{\lambda_i^{(k)}(x)\}$, $\lambda_i^{(k)}(x) \in \mathcal{A}|_{D_k}$ are piecewise continuous and as $\left\|K'_{k-1}(x_k) - K'_k(x_k)\right\| < 2\varepsilon$ so using lemmas 3.2 and 3.3 we can change these $\lambda_i(x)$ on every interval by unitarily equivalent ones to make

$$\operatorname{dist}(\lambda_i(x), \mathcal{A}) = \sup_k \left\|\lambda_i^{(k-1)}(x_k) - \lambda_i^{(k)}(x_k)\right\| < 2\varepsilon. \qquad \bullet$$

Lemma 3.5 *Let $K_1(x), K_2(x) \in M_n \otimes \mathbf{A}$ be piecewise continuous fields of operators with finite spectrum, $\|K_1(x) - K_2(x)\| < \varepsilon$, $K_1(x) \sim \operatorname{diag}\{\lambda_i(x)\}$, $K_2(x) \sim \operatorname{diag}\{\mu_i(x)\}$ with piecewise continuous fields $\lambda_i(x)$ and $\mu_i(x)$ such that $\operatorname{dist}(\mu_i(x), \mathcal{A}) < \delta$. Then there exist piecewise continuous fields of unitaries $u_i(x)$, piecewise continuous fields $\mu'_i(x)$ and a piecewise continuous operator field $K'_2(x) \sim \operatorname{diag}\{\mu'_i(x)\}$ such that*

i) $\|K'_2(x) - K_2(x)\| < \delta$;

ii) $\|u_i^*(x)\mu'_i(x)u_i(x) - \mu_i(x)\| < \delta$;

iii) $\|\lambda_i(x) - \mu'_i(x)\| < C\sqrt[4]{2\varepsilon + \delta}$;

iv) $\operatorname{dist}(\mu'_i(x), \mathcal{A}) < \delta$.

Proof. Take a point $x_0 \in X$ of continuity for $K_2(x)$. Let $D \supset x_0$ be an interval in X. By lemma 3.2 we can diagonalize the operator $K_2(x_0)$ so that its "eigenvalues" $\widetilde{\mu}_i(x_0)$ would satisfy the estimate

$$\|\widetilde{\mu}_i(x_0) - \lambda(x_0)\| < \varepsilon.$$

We can arbitrarily extend these "eigenvalues" to continuous fields $\widetilde{\mu}_i(x)$ so that in some neighborhood of the point x_0 one still has

$$\|\widetilde{\mu}_i(x) - \lambda_i(x)\| < \varepsilon.$$

By assumption there exist such unitaries $w_i \in A(x_0)$ that $w_i^* \widetilde{\mu}(x_0) w_i = \mu_i(x_0)$. Take a unitary extensions $w_i(x)$ of the elements w_i, $w_i(x_0) = w_i$ and choose a neighborhood of the point x_0 so that the estimate

$$\|w_i^*(x)\widetilde{\mu}_i(x)w_i(x) - \mu_i(x)\| < \delta$$

would hold in this neighborhood. Then we can get a division of X by intervals $D_k = [x_k; x_{k+1}]$ and piecewise continuous fields $\widetilde{\mu}_i(x)$, $\widetilde{\mu}_i(x)|_{D_k} = \widetilde{\mu}_i^{(k)}(x)$, $\widetilde{\mu}_i^{(k)}(x) \in \mathcal{A}|_{D_k}$ and piecewise continuous unitaries $w_i(x) = \{w_{i,k}(x)\}$, $w_{i,k}(x) \in \mathcal{A}|_{D_k}$ such that on D_k one has

$$\left\|\widetilde{\mu}_i^{(k)}(x) - \lambda_i(x)\right\| < \varepsilon \tag{3.6}$$

and

$$\left\|w_{i,k}^*(x)\widetilde{\mu}_i^{(k)}(x)w_{i,k}(x) - \mu_i(x)\right\| < \delta/2. \tag{3.7}$$

There exists also a piecewise continuous field of operators $\widetilde{K}_2(x)$ unitarily equivalent to the operator $\operatorname{diag}\{\widetilde{\mu}_i(x)\}$ and

$$\left\|\widetilde{K}_2(x) - K_2(x)\right\| < \delta/2.$$

It follows from (3.7) that at the point x_k one has

$$\left\|w_{i,k-1}^*(x_k)\widetilde{\mu}_i^{(k-1)}(x_k)w_{i,k-1}(x_k) - w_{i,k}^*(x_k)\widetilde{\mu}_i^{(k)}(x_k)w_{i,k}(x_k)\right\| < \delta. \tag{3.8}$$

It follows from (3.8) that there exist such unitaries $v_{i,k}(x_k) \in A(x_k)$ that

$$\left\|v_{i,k}^*(x_k)\widetilde{\mu}_i^{(k-1)}(x_k)v_{i,k}(x_k) - \widetilde{\mu}_i^{(k)}(x_k)\right\| < \delta.$$

By the lemma 3.3 we can find extensions of the unitaries $v_{i,k}(x_k)$ to some neighborhood of the point x_k such that due to (3.6) we have the estimate

$$\left\|v_{i,k}^*(x)\widetilde{\mu}_i^{(k)}(x)v_{i,k}(x) - \lambda_i(x)\right\| < C\sqrt[4]{2\varepsilon + \delta}.$$

Put

$$\overline{\mu}_i^{(k)}(x) = v_{i,k}(x)\widetilde{\mu}_i^{(k)}(x)v_{i,k}^*(x).$$

Then we have

$$\left\|\overline{\mu}_i^{(k)}(x) - \widetilde{\mu}_i^{(k-1)}(x)\right\| < \delta \tag{3.9}$$

and

$$\left\|\overline{\mu}_i^{(k)}(x) - \lambda_i(x)\right\| < C\sqrt[4]{2\varepsilon + \delta}.$$

Acting by induction and passing from D_{k-1} to D_k we change fields $\widetilde{\mu}_i^{(k)}(x)$ by $\overline{\mu}_i^{(k)}(x)$ on every D_k and so we obtain new piecewise continuous fields $\mu_i'(x) = \{\overline{\mu}_i^{(k)}(x)\}$ such that there exist piecewise continuous fields of unitaries $u_i(x)$ with

$$\|u_i^*(x)\mu_i'(x)u_i(x) - \mu_i(x)\| < \delta.$$

If we put $K_2'(x) = \mathrm{diag}\{u_i(x)\mu_i'(x)u_i^*(x)\}$ then we get validity of (i). Finally we conclude from (3.9) that $\mathrm{dist}(\mu_i'(x), \mathcal{A}) < \delta$. $\bullet$

Let now $K(x)$ be a strictly positive compact operator on the module $H_{\mathcal{A}}$. Take a sequence $\varepsilon_m > 0$ converging to zero. Due to its compactness of $K(x)$ by the lemma 3.4 one can find a sequence of piecewise continuous fields of diagonalizable finite rank operators $K_m(x) \in M_{n_m} \otimes \mathbf{A}$ with "eigenvalues" $\lambda_{i,m}(x) \in \mathcal{A}$ such that

i) $\|K_m(x) - K(x)\| < \varepsilon_m$,

ii) $K_m(x) \sim \mathrm{diag}\{\lambda_{i,m}(x)\}$ with $\mathrm{dist}(\lambda_{i,m}(x), \mathcal{A}) < \varepsilon_m$.

Now as

$$\|K_1(x) - K_2(x)\| \leq \|K_1(x) - K(x)\| + \|K(x) - K_2(x)\| < \varepsilon_1 + \varepsilon_2,$$

so by the lemma 3.5 we can find a piecewise continuous operator $K_2'(x) \sim \mathrm{diag}\{\lambda_{i,2}'(x)\}$ such that

$$\|K_2'(x) - K_2(x)\| < \varepsilon_2, \quad \left\|\lambda_{i,2}'(x) - \lambda_{i,2}(x)\right\| < \varepsilon_2, \quad \mathrm{dist}(\lambda_{i,2}'(x), \mathcal{A}) < \varepsilon_2,$$

and

$$\left\|\lambda_{i,2}'(x) - \lambda_{i,1}(x)\right\| < C\sqrt[4]{2(\varepsilon_1 + \varepsilon_2) + \varepsilon_2} < 2C\sqrt[4]{\varepsilon_1 + \varepsilon_2}.$$

Then as

$$\begin{aligned}\left\|K_{m-1}'(x) - K_m(x)\right\| &\leq \left\|K_{m-1}'(x) - K_{m-1}(x)\right\| + \|K_{m-1}(x) - K(x)\| \\ &+ \|K(x) - K_m(x)\| < 2\varepsilon_{m-1} + \varepsilon_m,\end{aligned}$$

so by induction we can find a sequence $K_m'(x) \sim \mathrm{diag}\{\lambda_{i,m}'(x)\}$ such that we have

$$\|K_m'(x) - K_m(x)\| < \varepsilon_m, \tag{3.10}$$

$$\left\|\lambda_{i,m}'(x) - \lambda_{i,m}(x)\right\| < \varepsilon_m,$$

$$\mathrm{dist}(\lambda_{i,m}'(x), \mathcal{A}) < \varepsilon_m, \tag{3.11}$$

$$\left\|\lambda_{i,m}'(x) - \lambda_{i,m-1}(x)\right\| < C\sqrt[4]{2(2\varepsilon_{m-1} + \varepsilon_m) + \varepsilon_m} < 2C\sqrt[4]{\varepsilon_{m-1} + \varepsilon_m}. \tag{3.12}$$

As by (3.10) as

$$\|K_m'(x) - K(x)\| \leq \|K_m'(x) - K_m(x)\| + \|K_m(x) - K(x)\| < 2\varepsilon_m,$$

so the sequence of operators $K'_m(x)$ converges (in norm) to the operator $K(x)$ and from (3.12) we see that for every i the sequences $\lambda'_{i,m}(x)$ are Cauchi sequences provided the numbers ε_m tend to zero fast enough, namely if the series $\sum_m \sqrt[4]{\varepsilon_{m-1} + \varepsilon_m}$ is convergent. Hence there exist the limits

$$\overline{\lambda}_i(x) = \lim_{m\to\infty} \lambda'_{i,m}(x)$$

and as by (3.11) $\mathrm{dist}(\lambda'_{i,m}(x), \mathcal{A})$ tends to zero, so $\overline{\lambda}_i(x) \in \mathcal{A}$. Show that these $\overline{\lambda}_i(x)$ are the "eigenvalues" for the operator $K(x)$. By the theorem 1.3 it follows from the estimate $\|K'_m(x) - K(x)\| < 2\varepsilon_m$ that there exist unitary operators $U_m \in M_{n_m} \otimes \mathbf{A}$ which map the " eigenvectors" $x_i \in H^*_{\mathbf{A}}$ of the operator $K(x)$ to "eigenvectors" of the operators $K'_m(x)$ such that $\|U^*_m K'_m U_m - K\| < 2\varepsilon_m$. Put $\widetilde{K}_m = U^*_m K'_m U_m \in M_{n_m} \otimes \mathbf{A}$; $\widetilde{K}_m \to K$. Then one has

$$\widetilde{K}_m x_i = x_i \lambda'_{i,m}. \tag{3.13}$$

Taking limit in (3.13) we obtain $K x_i = x_i \overline{\lambda}_i$. •

4 Example: continuous field of rotation algebras

Finally we give an example of a C^*-algebra without property of weak diagonalization. Let $X = [a, b]$ be an interval of the real line containing an integer, say 1. Let S be a circle and let $C(X \times S)$ be the C^*-algebra of continuous functions on the cylinder. Let α be the action of the group $\mathbf{Z}$ of integers on this algebra defined by

$$(\alpha(n)f)(x, t) = f(x, t + nx), \tag{4.1}$$

where $f(x, t) \in C(X \times S)$, $x \in X$, $t \in S$, $n \in \mathbf{Z}$. Denote by A_X the crossed product

$$C(X \times S) \rtimes_\alpha \mathbf{Z}.$$

By [23],[5] the algebra A_X is a continuous fields of rotation algebras $A_x = A(x)$ over an interval. Notice that the continuous field $(A(x), X, A_X)$ is a continuous field of tracial C^*-algebras and for irrational x the algebras $A(x)$ are of real rank zero and topological rank one. Moreover, for each $x \in X$ the algebra $A(x)$ has the property of weak diagonalization. On the other hand the unitary group of $A(x)$ is not connected, but the obstruction lies not here. Notice that $A_1 \cong C(\mathbf{T}^2)$ is the commutative C^*-algebra of continuous functions on a torus, hence unlike the irrational case it is not of real rank zero and the map $K_0(A(1)) \longrightarrow K_0(B(1))$ is not a monomorphism. There exists in $M_2 \otimes A_1$ a projection which gives the Bott generator for $K^0(\mathbf{T}^2)$. This projection can be extended [22],[17] to a continuous

field of projections $p(x)$ in a neighborhood of the point $1 \in X$ so that for the standard trace τ_x on A_X we have

$$\tau_x(p(x)) = 1 + x. \tag{4.2}$$

We can diagonalize this field of projections in the direct integral of type II_1 factors, $p(x) \sim \mathrm{diag}\{\lambda_1(x), \lambda_2(x)\}$ with $\lambda_1(x) \geq \lambda_2(x)$. Then for $x = 1$ we should have $\lambda_1(1) = 1$, $\lambda_2(1) = 0$. For any x the "eigenvalues" $\lambda_i(x)$, $i = 1, 2$, should be projections. If these "eigenvalues" are continuous fields in A_X then for all x $\lambda_1(x) = 1$, $\lambda_2(x) = 0$, hence

$$\tau_x(\mathrm{diag}\{\lambda_1(x), \lambda_2(x)\}) = 1.$$

But as the trace is invariant under unitary equivalence, so it contradicts (4.2), therefore the C^*-algebra A_X has no property of weak diagonalization.

On the other hand as the algebras $A(x)$ satisfy all conditions of the theorem 3.1 for a dense set of irrational points in X, so slightly modifying our proof we can for given continuous field of operators $K(x)$ and for any $\varepsilon > 0$ find a subset $X_\varepsilon \subset X$ so that the measure of $X \setminus X_\varepsilon$ is less than ε and $K(x)$ is diagonalizable on X_ε with continuous "eigenvalues".

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V. M. Manuilov
Dept. of Mech. and Math.,
Moscow State University,
Moscow, 119899, RUSSIA
e-mail: manuilov@mech.math.msu.su